Coupling Front Tracking and Wavelet Techniques for Fast Time Domain Simulations

Costas D. Sarris and Linda P. B. Katehi
Radiation Laboratory, Department of Electrical Engineering and Computer Science
University of Michigan, Ann Arbor, 48109-2122, MI

Abstract— **The main attractive feature of wavelet-based, time-domain techniques is the simple implementation of adaptive meshing, through the application of a thresholding procedure to eliminate wavelet coefficients that attain relatively insignificant values, at a limited compromise of accuracy. However, little attention has been devoted so far to the investigation of computational costs and accuracy trade-offs in order to obtain thresholding-related operation savings. This paper presents an efficient implementation of thresholding applied to a non-linear problem and reports significant execution time savings compared to the conventional FDTD technique, that the application of the proposed method has led to.**

Keywords—**Multiresolution Analysis, MRTD, FDTD, wave propagation, front tracking.**

I. Introduction

The use of Multiresolution Analysis (MRA) concepts for the development of time-domain numerical schemes has been recently studied in significant extent and a wide range of applications has already been demonstrated [1]-[7]. A motivating force for this research activity is the fact that wavelet-based methods provide the most natural framework for the implementation of adaptive grids, dynamically following local variations and singularities of solutions to partial differential equations. In particular, it can be proven that the decay of wavelet expansion coefficients of a square integrable function depends on the local smoothness of the latter [6]. Hence, significant wavelet values are expected at space-time regions, where high variations in the numerical solution evolve. In this sense, sparing the arithmetic operations on wavelet coefficients below a certain threshold -small enough according to accuracy requirements- amounts to imposing coarse gridding conditions at those regions, while allowing for a denser mesh at parts of the domain where the solution varies less smoothly.

Several approaches to wavelet based mesh refinement have been presented in the literature. In [4], [5], Haar wavelets were used to selectively refine the resolution of an underlying FDTD scheme at parts of the domain where this seemed to be a priori necessary, in a static sense. However, the resulting method was strictly equivalent to a subgridded FDTD and presented obvious accuracy disadvantages, since it was not enhanced by the interpolatory operations that subgridded FDTD typically employs [8]. In general, the necessity to incorporate wavelets into a time-domain simulation technique is always related to dynamic rather than static subgridding, since the latter -having been extensively studied in the literature- can be nowadays efficiently implemented by several existing engines.

On the contrary, adaptive, wavelet-based meshing was introduced in [6] and applied to electromagnetic structures in [2], [7], always in conjunction with high order basis functions, these being either Daubechies or B-splines. In both cases, the complexity of the proposed scheme made clear that "the construction of wavelet-based discretizations with a robustness and flexibility comparable to FDTD is still a challenging task" [7]. This paper meets this challenge by building up a two-level scheme on the simplest wavelet basis, the Haar basis and attempting a simple and explicit implementation of an algorithm for the thresholding of wavelet coefficients that is based on ideas originally related to shock-wave problems of computational fluid dynamics. Execution time measurements for the algorithm as applied to a nonlinear optics problem, show that this procedure can actually lead to faster-than-FDTD simulations.

II. Adaptive Haar Wavelet Simulation of Pulse Compression in an Optical Fiber Filter

As a vehicle for the demonstration of the algorithms developed in this study, the propagation of an optical pulse through a fiber filter and its gradual compression is simulated. This nonlinear phenomenon is based on a self-phase modulation (SPM) induced negative dispersion that the pulse experiences along the fiber and was originally reported and studied in [10]. It is noted that the refractive index in the fiber assumes the form : $n(z) = n_0 + n_1 \cos(2\beta_0 z) + n_2|E|^2$, where the cosinusoidal term is due to a periodic structure (grating) written within the core of the fiber. Numerical modeling of this case was also pursued in [11], by means of the Battle-Lemarie cubic spline based S-MRTD technique, resulting in execution time that was reported to be larger than FDTD by a factor of 1.5. In this work, a one wavelet level Haar MRTD scheme is used, for the purpose of estimating the improvement in computational performance that exclusively originates from the dynamic adaptivity of MRTD rather than the high order of an underlying scaling basis.

A. Formulation of Haar MRTD scheme

The electric field in the fiber is decomposed in forward and backward propagating waves as :

$$E(z,t) = E_F(z,t)\, e^{j(\beta z-\omega t)} + E_B(z,t)\, e^{-j(\beta z+\omega t)}. \quad (1)$$

Substituting this expression into Maxwell's equations and discarding terms with spatial variation faster than $e^{j2\beta z}$, the following system of equations is deduced :

$$\frac{\partial E_F}{\partial z} + \frac{n_0}{c}\frac{\partial E_F}{\partial t} = j\kappa E_B\, e^{-2j\Delta\beta z} + j\gamma\left(|E_F|^2 + 2|E_B|^2\right) E_F \quad (2)$$

$$\frac{\partial E_B}{\partial z} - \frac{n_0}{c}\frac{\partial E_B}{\partial t} = -j\kappa E_F\, e^{2j\Delta\beta z} - j\gamma\left(|E_B|^2 + 2|E_F|^2\right) E_B \quad (3)$$

with $\gamma = \pi n_2/\lambda$, $\kappa = \pi n_1/\lambda$ and $\Delta\beta(\omega) = n_0(\omega)\omega/c - 2\pi n_0/\lambda_0$ and λ_0 the free-space wavelength that satisfies the Bragg condition for the grating.

While the FDTD update equations for the system of (2), (3) can be retrieved from [11], Haar MRTD equations are derived here, by first expanding both forward and backward fields in terms of Haar scaling $\phi_m(z) = \phi(z/\Delta z - m)$, wavelet $\psi_m(z) = \psi(z/\Delta z - m)$ functions in space and pulse functions $h_k(t) = h(t/\Delta t - k)$ in time, following the definitions of [12] for those:

$$E_x(z,t) = \sum_{m,k=-\infty}^{\infty} \left({}_kE_m^{x,\phi}\, \phi_m(x) + {}_kE_m^{x,\psi}\, \psi_m(x) \right) h_k(t) \quad (4)$$

where $x = F, B$. Upon substitution of (4) into (2), (3) and application of the Method of Moments as explained in [1], the system of Haar MRTD update equations is formulated. As an example, the update equation for the m-th cell value of $E^{F,\phi}$ at the k-th time step reads :

$$\begin{aligned} {}_{k+1}E_m^{F,\phi} &= {}_{k-1}E_m^{F,\phi} - s\Big\{ {}_kE_{m+1}^{F,\phi} - {}_kE_{m-1}^{F,\phi} + \\ &+ \left({}_kE_{m+1}^{F,\psi} - 2\,{}_kE_m^{F,\psi} + {}_kE_{m-1}^{F,\psi} \right) \Big\} \\ &+ 2j\kappa s\Delta z e^{-2j\Delta\beta(m+1/2)\Delta z} \{ \mathbf{Sa}(\Delta\beta\Delta z) \\ &\times {}_kE_m^{B,\phi} + j\mathbf{Sa}'(\Delta\beta\Delta z/2)\,{}_kE_m^{B,\psi} \} \\ &+ 2j\gamma s\Delta z \left\{ {}_k\mathcal{A}_m\, {}_kE_m^{F,\phi} + {}_k\mathcal{B}_m\, {}_kE_m^{F,\psi} \right\}, \end{aligned} \quad (5)$$

with $\mathbf{Sa}(w) = \sin w/w$, $\mathbf{Sa}'(w) = \sin^2 w/w$ and $s = c\Delta t/\Delta z$, being the FDTD CFL (Courant -Friedrichs-Lewy) number. Furthermore,

$$\begin{aligned} {}_k\mathcal{A}_m &= |\,{}_kE_m^{F,\phi}|^2 + |\,{}_kE_m^{F,\psi}|^2 + 2\left(|\,{}_kE_m^{B,\phi}|^2 + |\,{}_kE_m^{B,\psi}|^2 \right) \\ {}_k\mathcal{B}_m &= 1/2 \left(|\,{}_kE_m^{F,\phi} + {}_kE_m^{F,\psi}|^2 - |\,{}_kE_m^{F,\phi} - {}_kE_m^{F,\psi}|^2 \right) \\ &+ |\,{}_kE_m^{B,\phi} + {}_kE_m^{B,\psi}|^2 - |\,{}_kE_m^{B,\phi} - {}_kE_m^{B,\psi}|^2 \end{aligned} \quad (6)$$

Similar equations hold for the rest of the unknowns, namely ${}_kE_m^{F,\psi}$, ${}_kE_m^{B,\phi}$, ${}_kE_m^{B,\psi}$. It is worth mentioning that operations in (5) can be readily split in scaling- and wavelet-related ones, thus facilitating their adaptive application.

B. Boundary Conditions

Assuming an optical fiber that extends from $z = 0$ to $z = L$, the following boundary conditions are imposed :

$$E_F(0,t) = \sqrt{\frac{A}{2}}\,(1+j)\,e^{-t^2/(2\alpha^2)}, \;\; E_B(L,t) = 0 \quad (7)$$

by local modification of the update equations, based on the explicit enforcement of those conditions (without extrapolations or interpolations as proposed in the past [3]). For example, if $z = 0$ belongs to the M−th cell of the domain, and $g_k = E_F(0, k\Delta t)$ is the discrete-time sample of the excitation function at the k-th step, then :

$${}_{k+1}E_M^{F,\phi} = 0.5\,(g_{k+1} + \lambda_{k+1})\,, {}_{k+1}E_M^{F,\psi} = 0.5\,(g_{k+1} - \lambda_{k+1}) \quad (8)$$

where $\lambda_k = {}_kE_M^{F,\phi} - {}_kE_M^{F,\psi}$ is explicitly updated via the equation :

$$\begin{aligned} \lambda_{k+1} &= \lambda_{k-1} \\ &- s\left({}_kE_{M+1}^{F,\phi} + {}_kE_{M+1}^{F,\psi} - {}_kE_M^{F,\phi} - {}_kE_M^{F,\psi} \right) \\ &+ j\gamma s\Delta z \left(|\lambda_k|^2 + 2|\,{}_kE_M^{B,\phi} - {}_kE_M^{B,\psi}|^2 \right) \lambda_k \end{aligned} \quad (9)$$

Similar expressions are derived for the application of the hard boundary condition on the backward wave at the terminal cell of the MRTD domain.

Finally, absorbing boundary conditions for backward and forward waves are imposed at $z = 0,\ L$, via matched layer absorbers. The absorbers are implemented as in [11], by expanding their quadratically varying conductivities in Haar scaling functions. A maximum conductivity $\sigma_{max} = 0.1$ S/m is used in all subsequent simulations.

C. Validation

For validation purposes, results of a non-adaptive Haar MRTD code are compared to FDTD. The parameters of the fiber are normalized, with respect to the length L of the filter, as follows : $\kappa L = 4$, $\Delta\beta L = 12$, $\gamma L = 2/3$. The time step is $\Delta t = 0.003125 n_0 L/(10c)$ and two cases for the FDTD cell size are considered : $\Delta z = 0.001L$ and $\Delta z = 0.002L$. Respectively, scaling cell sizes for MRTD are chosen to be $\Delta z = 0.002L$ and $\Delta z = 0.004L$, as the introduction of one wavelet level refines the resolution by a factor of two, thus allowing for the reduction of the number of cells in half. Matched layers of 500 and 250 cells terminate the FDTD and MRTD meshes respectively, with maximum conductivity $\sigma_0 = 0.1$ S/m. Simulation data for both aforementioned cases are provided in Table I. Execution times for the relevant FDTD and MRTD cases over 12,000 time steps (on a Sun Ultra 80 workstation at 500 MHz) reveal that the non-adaptive MRTD code is slower than FDTD by a factor of 11 - 16 %. The reason for this expected slowdown is that the factor of increase in operations per cell between MRTD and FDTD is greater than the ratio of FDTD to MRTD cells (equal to two). The two methods agree well on the peak of the transmitted intensity (which corresponds to the forward wave intensity at the end of the fiber). The slight difference can be attributed to the fact that in the MRTD absorber, the conductivity was assumed constant within each scaling cell and therefore it varied less smoothly than the corresponding FDTD absorber conductivity. Moreover, Fig. 1 depicts the pulse compression along the fiber, since it includes pulse waveforms (extracted via 1000 cell FDTD and 500 cell MRTD), as probed at the beginning, the middle and the end of the fiber filter. Again, excellent correlation between the two methods (FDTD and MRTD) is demonstrated. In consequence, the non-adaptive MRTD results can be henceforth used as a measure for accuracy estimation of adaptive MRTD algorithms.

TABLE I
VALIDATION DATA FOR NON-ADAPTIVE MRTD CODE

Parameter	FDTD, I	MRTD, I	FDTD, II	MRTD, II
cells	1000	500	500	250
CPU time[sec]	43.25	48.23	25.55	29.56
$max\|E_F(z=L)\|^2$	10.6042	10.6635	10.3541	10.4283

Finally, Figs. 2, 3 show snapshots of the forward field intensity and the magnitude squared of the forward field wavelet coefficients plotted in space-time coordinates. Evidently, the evolution of both the wavelet coefficients and the field itself takes place along the characteristic line of forward field propagation, $z = ct$. A similar pattern for wavelet behavior was obtained in [7], where non-uniform multiconductor transmission

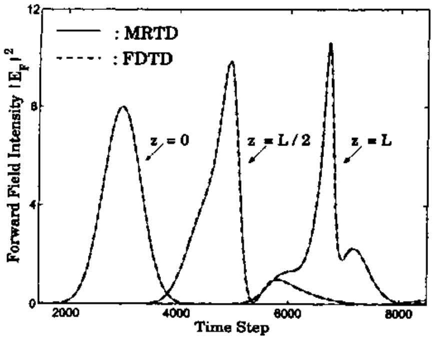

Fig. 1. MRTD and FDTD results for the forward field intensity at the beginning, the middle and the end of the optical fiber filter (1000 grid points).

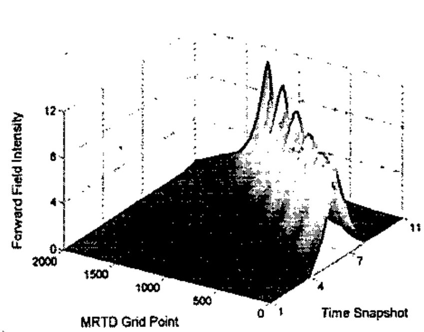

Fig. 2. Spatio-temporal propagation and compression of the nonlinear optical pulse (forward field intensity is shown).

line equations were solved via a biorthogonal wavelet basis. In the following, this observation is utilized for the development of a computationally efficient approach to the problem of thresholding of wavelet coefficients.

D. Implementation and Performance of Adaptive MRTD

Regarding the development of adaptive algorithms, two questions are addressed in this work: First, the application of thresholding with as few operations (namely checks on wavelet coefficients being absolutely above or below a threshold ϵ) as possible and second, the stable exploitation of the compressed representation of the solution for the reduction in update operations at each time step. For this purpose, the following method is adopted [2], [6], [7], for both forward and backward propagating wave arrays:

- All cells are initialized as being above threshold (referred to as *active*).
- *Scaling coefficients* are updated throughout the mesh, along with active wavelet terms. In all operations, only scaling and active wavelet terms are taken into account.
- At each time step, the magnitudes of wavelet terms at cells that are designated as active are compared to an absolute threshold ϵ. The corresponding cells remain active only if their wavelets are above the threshold.
- The region of the active cells is extended to all nearest neighbors of the latter. Since only one wavelet level is used here, this implies cells that are immediately to the left or to the right of cells on the border of active regions.

Hence, thresholding checks and update operations are limited to a subset of the field coefficients. Note that the use of the Haar basis and a single wavelet level scheme, keeps the implementation of this adaptive algorithm relatively simple and readily expandable to three dimensions. On the other hand, the complexity of numerical solvers based on higher order basis functions and multiple wavelet levels has regularly undermined the potential of adaptivity to yield execution times better than FDTD. Furthermore, the frequency of thresholding checks is implicitly dependent on the CFL number s of the simulation, which effectively determines the maximum number of cells that a front may move through in a single time step. Approximately, a thresholding check window should be at most equal to $1/s$, with the previously described procedure strictly corresponding to the value of $s = 1$. Physically, the thresholding algorithm, that was first introduced for the numerical solution of shock wave problems in [6], assumes the evolution of wavelet coefficients along wavefronts defined from the characteristics of a given problem. Then, the purpose of adding pivot elements, to extend the domain of active coefficients, is actually the tracking of these wavefronts as they move throughout the computational domain.

To this end, the second case that was presented in section II-C is repeated here by using several absolute thresholds and CPU time measurements are carried out. Also, three thresholding windows are investigated, with the outlined algorithm being applied every one, two and four time steps respectively. Although the latter was slightly larger than $1/s$, it produced satisfactory (accuracy-wise) results. In all cases, thresholds up to 0.001 led to simulations that clearly resolved the pulse compression and suffered from errors (with respect to the unthresholded MRTD, which is used as a reference solution) of less than 1%.

Fig. 4 depicts forward field intensity waveforms sampled at $z = L$, for thresholds 0.1, 10^{-4} and 10^{-7}. The last two are in good agreement both with each other and with the previously presented FDTD and MRTD results, while the first suffers from significant numerical errors, that demonstrate themselves as a ripple corrupting the pattern of the waveform. More explicitly, CPU economy with respect to FDTD and error in the peak transmitted forward field intensity are plotted (with respect to the absolute threshold ϵ), in Figs. 5, 6. It is thus shown, that the adaptive MRTD code can extract the solution to this problem, at a CPU time reduced (compared to FDTD) by a factor close to 30%, with errors limited at the order of 0.1%.

Thresholding operations in this problem represented a worst-case scenario for the computational overhead that the adaptive algorithm may bring about, for the following reasons : First, the geometry was a one-dimensional one, a significant part of which was almost throughout the simulation occupied by the propagating pulse and second, these operations involved complex numbers and nonlinear terms. Therefore, the fact that an accelerated performance of adaptive MRTD (with respect to FDTD) was achieved is important and demonstrates the potential of the algorithm for larger geometries.

III. Conclusions

Based on the study of a nonlinear pulse compression by an optical fiber filter, this paper demonstrated Haar wavelet based simulations with adaptive meshing that achieved better-than-FDTD execution times. To the extent of the authors' knowledge this is the first time when an adaptive, wavelet-based code

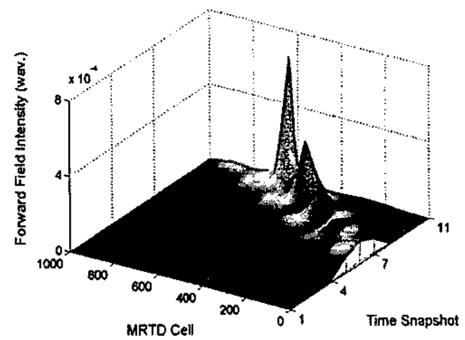

Fig. 3. Spatio-temporal evolution of forward field wavelet coefficients (square of magnitude of wavelet coefficients is shown).

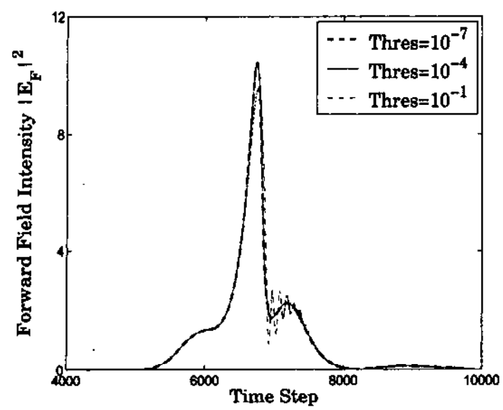

Fig. 4. Trasmitted forward field intensity for different thresholds, with thresholding applied every four time steps.

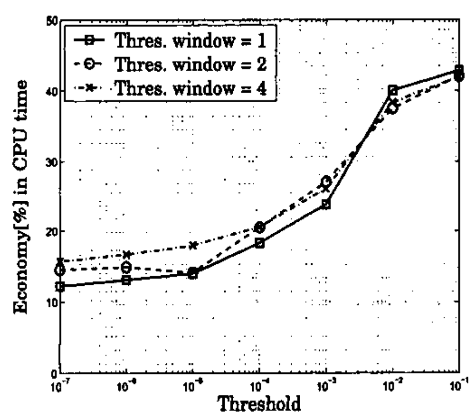

Fig. 5. CPU time economy for the adaptive MRTD code, with respect to FDTD, for absolute thresholds from 10^{-7} to 0.1 applied to Haar MRTD.

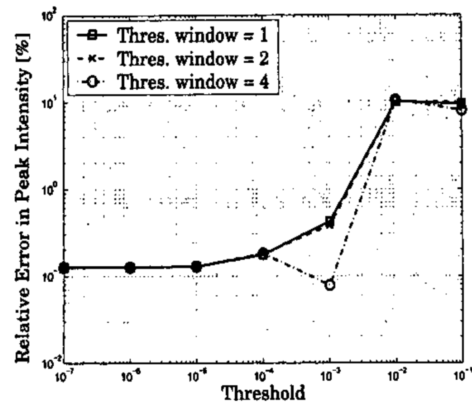

Fig. 6. Relative error [%] in the peak intensity of the transmitted forward field for adaptive Haar MRTD.

surpasses the efficiency of the conventional FDTD, not only in terms of memory but also in terms of execution time requirements. The satisfactory performance of the proposed technique stems from its relative simplicity that allows for an efficient implentation of its two components : thresholding tests of wavelet coefficients and operation savings while performing updates of field arrays.

IV. Acknowledgement

This work has been supported by the U.S. Army CECOM through a CHSSI/HPCMO project for "Efficient Numerical Solutions to Large Scale Tactical Communication Problems" (CEN-5).

References

[1] M. Krumpholz, L.P.B. Katehi, "MRTD: New Time Domain Schemes Based on Multiresolution Analysis", *IEEE Trans. Microwave Theory and Techniques*, vol.44, no.4, pp.555-561, April 1996.

[2] W. Werthen and I. Wolff, "A novel wavelet based time domain simulation approach", *IEEE Microwave and Guided Wave Lett.*, vol. 6, no. 12, pp. 438-440, Dec. 1996.

[3] M. Fujii and W. J. R. Hoefer, "A three-dimensional Haar wavelet-based multi-resolution analysis similar to the 3-D FDTD method - derivation and application", *IEEE Trans. Microwave Theory Tech.*, vol. 46, no. 12, pp. 2463-2475, Dec. 1998.

[4] T. Dogaru, L. Carin, "Multiresolution time-domain analysis of scattering from a rough dielectric surface ", *Radio Sci.*, vol. 35, pp. 1279-1292, Nov.-Dec. 2000.

[5] G. Carat, R. Gillard, J. Citerne, J. Wiart, "An efficient analysis of planar microwave circuits using a DWT-based Haar MRTD scheme ", *IEEE Trans. Microwave Theory and Techniques*, vol. 48, no. 12, pp. 2261-2270, Dec. 2000.

[6] E. Bacry, S. Mallat, G. Papanicolaou, "A Wavelet Based Space-Time Adaptive Numerical Method for Partial Differential Equations", *Mathematical Modelling and Numerical Analysis, 26, n. 7, p. 793 (1992).*

[7] S. Grivet-Talocia, "Adaptive Transient Solution of Nonuniform Multiconductor Transmission Lines Using Wavelets", *IEEE Trans. Antennas Prop.*, AP-48, no.10, October 2000, pp. 1563-1573.

[8] M. Okoniewski, E. Okoniewska, M. A. Stuchly, "Three Dimensional Subgridding Algorithm for FDTD", *IEEE Trans. Antennas Propagat.*, vol. 45, no.3, March 1997, pp. 422-429.

[9] E.Tentzeris, R.Robertson, A.Cangellaris and L.P.B.Katehi, "Space- and Time- Adaptive Gridding Using MRTD", *1997 IEEE MTT-S International Microwave Symposium Digest*, pp.337-340.

[10] H.G. Winful, "Pulse compression in optical fiber filters", *Appl. Phys. Lett.*, vol. 46, pp. 527-529, Mar. 1985.

[11] M. Krumpholz, H.G. Winful, L.P.B. Katehi, 'Nonlinear Time-Domain Modeling by Multiresolution Time Domain (MRTD)", *IEEE Trans. Microwave Theory and Techniques*, vol.45, no.3, pp.385-393, March 1997.

[12] C.D. Sarris, L.P.B. Katehi, "Fundamental Gridding Related Dispersion Effects in MRTD Schemes", *to appear in IEEE Trans. Microwave Theory and Techniques*, Dec. 2001.